

ON MONODROMY REPRESENTATIONS IN DENHAM-SUCIU FIBRATIONS

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ABSTRACT. We study the monodromy representation corresponding to a fibration introduced by G. Denham and A. Suciu [5], which involves polyhedral products given in Definition 2.2. Algebraic and geometric descriptions for these monodromy representations are given. In particular, we study the case of a product of two finite cyclic groups and obtain representations into $\text{Out}(F_n)$ and $SL_n(\mathbb{Z})$. We give algebraic descriptions of monodromy for the case of a product of any two finite groups. Finally we give a geometric description for monodromy representations of a product of 2 or more finite groups to $\text{Out}(F_n)$, as well as some algebraic properties. The geometric description does not rely on choosing a basis for the fundamental group of the fibre in terms of commutators, hence avoids this delicate question.

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1. INTRODUCTION

Let (X, A) be a pair of spaces and let K be a simplicial complex with n vertices. A new topological space can be constructed using the pair (X, A) and K , called a *polyhedral product* and denoted by $Z_K(X, A) \subset X^n$ (see Definition 2.2). Polyhedral products are actively studied and stand at the foundations of the field of *toric topology*, see for example work of A. Bahri, M. Bendersky, F. Cohen and S. Gitler

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[1] for an introduction, or work of V. Buchstaber and T. Panov [3] for a survey of toric topology. A short introduction on polyhedral products is given in Section 2.

Given a locally trivial fibration $f : E \rightarrow B$ with fibre F , there is an action of the fundamental group of B on the fundamental group of the fibre F and consequently the first homology of F . This action gives rise to a representation called the monodromy representation. One natural use of this representation is calculating the homology of the total space E using the Serre spectral sequence, when the fundamental group of the base space B acts non-trivially on the homology of F . In that case the homology of F is a non-trivial module over the group ring $\mathbb{Z}\pi_1(B)$.

Let G be a topological group. Let BG denote the classifying space of G and EG denote a contractible space on which G acts freely and properly discontinuously. The projection $EG \rightarrow BG = EG/G$ is then a bundle projection. In particular, if G is a finite discrete group, then EG is the universal cover of BG . G. Denham and A. Suciu [5, Lemma 2.3.2] gave a natural fibration relating the polyhedral product for the pair $(BG, *)$, where $*$ is the basepoint of BG , and the polyhedral product for (EG, G) . That is, for a simplicial complex K with n vertices the polyhedral product $Z_K(BG, *)$ fibres over the product $(BG)^n$ as follows

$$(1.1) \quad Z_K(EG, G) \rightarrow (EG)^n \times_{G^n} Z_K(EG, G) \rightarrow (BG)^n,$$

where $(EG)^n \times_{G^n} Z_K(EG, G) \simeq Z_K(BG, *)$. The group G acts on the space $Z_K(EG, G) \subset (EG)^n$ coordinate-wise, thus there is an action of G^n on the fibre $Z_K(EG, G)$. This fibration generalizes previous constructions in work of M. Davis and T. Januszkiewicz [4] and work of V. Buchstaber and T. Panov [2]. In particular, it originates from the Davis-Januszkiewicz space defined by

$$\mathcal{DJ}(K) = E(S^1)^n \times_{(S^1)^n} Z_K(D^2, S^1),$$

and generalizes the result of V. Buchstaber and T. Panov that the homotopy fibre of the inclusion

$$Z_K(BS^1, *) \hookrightarrow (BS^1)^n$$

is homotopy equivalent to $Z_K(ES^1, S^1)$.

This paper studies the monodromy representation of the natural fibration in equation 1.1 introduced by G. Denham and A. Suciu [5], where all the spaces are polyhedral products of special pairs of spaces. The monodromy action is then described naturally using the geometry of the fibre, which arises from properties of polyhedral products. We will use convenient models for the homotopy type of the pairs (X, A) to achieve this. In certain cases we will be able to give precise algebraic descriptions of the action.

The fibration in equation 1.1 can be slightly generalized if we allow for the group G to vary in each coordinate. In a similar way, one can define a polyhedral product for a sequence of pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^n$ and a simplicial complex K , and denote it by $Z_K(\underline{X}, \underline{A})$, see Definition 2.2. Two such sequences of pairs are $(\underline{BG}, *) = \{(BG_i, *)\}_{i=1}^n$ and $(\underline{EG}, \underline{G}) = \{(EG_i, G_i)\}_{i=1}^n$. Then there is a bundle

$$Z_K(\underline{EG}, \underline{G}) \rightarrow \left(\prod_{i=1}^n EG_i \right) \times_{(\prod_{i=1}^n G_i)} Z_K(\underline{EG}, \underline{G}) \rightarrow \prod_{i=1}^n BG_i,$$

where $(\prod_{i=1}^n EG_i) \times_{(\prod_{i=1}^n G_i)} Z_K(\underline{EG}, \underline{G}) \simeq Z_K(\underline{BG}, *)$. Therefore, we can rewrite the fibration in (1.1) as follows

$$(1.2) \quad Z_K(\underline{EG}, \underline{G}) \rightarrow Z_K(\underline{BG}, *) \rightarrow \prod_{i=1}^n BG_i.$$

Similarly, $G_1 \times \cdots \times G_n$ acts on the fibre $Z_K(\underline{EG}, \underline{G}) \subset EG_1 \times \cdots \times EG_n$ coordinate-wise. We will refer to this fibration as the *Denham–Suciu fibration*.

Now suppose that G_1, \dots, G_n are discrete groups. One can study the action of the fundamental group of the base space on the fundamental group of the fibre, namely the action of $G_1 \times \cdots \times G_n$ on $\pi_1(Z_K(\underline{EG}, \underline{G}))$. A natural question is also determining the structure of the first homology group of $Z_K(\underline{EG}, \underline{G})$ as a module over the group ring $\mathbb{Z}[G_1 \times \cdots \times G_n]$.

There are cases where the fundamental group of the fibre can be given explicitly. In particular, if $K = K_0$ is the zero simplicial complex consisting of only n vertices and no edges, and if G_1, \dots, G_n are finite discrete groups, then it is shown in [6, Theorem 3.8] that $\pi_1(Z_{K_0}(\underline{EG}, \underline{G})) \cong F_{N_n}$, the free group of rank N_n . The natural number N_n is shown in [6, Corollary 3.7] to be

$$(1.3) \quad N_n = (n-1) \prod_{i=1}^n m_i - \sum_{i=1}^n (\prod_{j \neq i} m_j) + 1,$$

where $m_i = |G_i|$. Note that the rank of the free group in this case depends only on the order of the groups G_i . Moreover, for the special case of K_0 all the spaces in the Denham–Suciu fibration are Eilenberg–Mac Lane spaces, see [6, Theorem 1.1], therefore there is a short exact sequence of groups

$$1 \rightarrow F_{N_n} \rightarrow \pi_1(Z_{K_0}(\underline{BG}, *)) \rightarrow \prod_{i=1}^n G_i \rightarrow 1.$$

By definition of polyhedral products we have $Z_{K_0}(\underline{BG}, *) = BG_1 \vee \cdots \vee BG_n$, see Example 2.3. Hence, we get

$$1 \rightarrow F_{N_n} \rightarrow G_1 * \cdots * G_n \rightarrow \prod_{i=1}^n G_i \rightarrow 1,$$

where $G * H$ denotes the free product of the groups G and H . Note that F_{N_n} is generated by commutators in the free product $G_1 * \cdots * G_n$. In fact this is what makes monodromy action a delicate question. There is no obvious “nice” basis for the free group F_{N_n} in terms of commutators that makes the algebraic computations of the monodromy representation accessible. To avoid this problem, we replace the pairs (EG_i, G_i) with $([0, 1], F_i)$, where F_i is a subset of the unit interval $[0, 1]$ with the same cardinality as G_i , and give the monodromy action geometrically for the general case, see Section 3. This is possible since up to homotopy polyhedral products depend only on the relative homotopy type of the pairs (X, A) .

For F_n the free group of rank n let $\text{Aut}(F_n)$ and $\text{Inn}(F_n)$ denote the group of automorphisms and the group of inner automorphisms of F_n , respectively. Let $\text{Out}(F_n) := \text{Aut}(F_n)/\text{Inn}(F_n)$ denote the group of outer automorphisms of F_n .

Then the monodromy action for the finite discrete groups G_1, \dots, G_n and the zero simplicial complex K_0 is given by the representation

$$\rho_{K_0} : G_1 \times \cdots \times G_n \rightarrow \text{Out}(F_{N_n}),$$

see Section 3.3.

Similarly, in [6, Theorem 1.1] it is shown that $Z_K(\underline{BG}, *)$ is an Eilenberg–Mac Lane space if and only if K is a *flag complex* (see Definition 2.6), and if K is a flag complex there is also a representation

$$\rho_K : G_1 \times \cdots \times G_n \rightarrow \text{Out}(\pi_1(Z_K(\underline{EG}, \underline{G}))),$$

see Section 3.3. Note that the zero simplicial complex K_0 is a special case of a flag complex. The computations will be restricted to K_0 , and in the last section we will discuss the representations for other choices of K .

We obtain the following results regarding monodromy representations.

Proposition 1.1. *Let $G = \{g_1, \dots, g_m\}$ and $H = \{h_1, \dots, h_n\}$ be two finite discrete groups of order m and n , respectively. Then the monodromy action is given by the faithful representation*

$$\varphi : G \times H \rightarrow \text{Out}(F_k)$$

where for any $t \in G \times H$, the image $\varphi(t) = \varphi_t$ is given by the following equations

$$(1.4) \quad \begin{aligned} \varphi_{g_k}([g_i, h_j]) &= g_k[g_i, h_j]g_k^{-1} = [g_k g_i, h_j][h_j, g_k] \\ \varphi_{h_k}([g_i, h_j]) &= h_k[g_i, h_j]h_k^{-1} = [h_k, g_i][g_i, h_k h_j], \end{aligned}$$

and $k = (m-1)(n-1)$.

As a special case we work out the case of two finite cyclic groups explicitly.

Theorem 1.2. *Let $G_1 = C_n$ and $G_2 = C_m$ be two finite cyclic groups of order n and m , respectively. Then the monodromy action is given by a representation*

$$C_n \times C_m \rightarrow \text{Out}(F_k),$$

where $k = (n-1)(m-1)$, which gives a faithful representation

$$C_n \times C_m \rightarrow SL_k(\mathbb{Z}),$$

given by equations 4.1 and 4.2.

The method presented in this paper applies to any finite collection of finite discrete groups G_1, \dots, G_n . Let $\text{gr}_*(G)$ denote the Lie algebra associated to the descending central series of the group G . Finally, we give some properties of the representation

$$G_1 \times \cdots \times G_k \rightarrow \text{Out}(F_{N_k})$$

for finite discrete groups.

Lemma 1.3. *Let $\{G_i\}_{i=1}^n$ be a collection of finite discrete groups and K_0 be the 0-simplicial complex on n vertices. Let $\rho : \prod_{i=1}^n G_i \rightarrow \text{Out}(F_N)$ be the monodromy representation where F_N is isomorphic to the kernel of the projection*

$$p : G_1 * \cdots * G_n \rightarrow \prod_{i=1}^n G_i.$$

Then the following properties of ρ hold:

- (1) There is a choice of a generating set for F_N that consists of elements of the form

$$f = [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]] \in \Gamma^k(G_1 * \dots * G_n)$$

such that $g_{i_j} \in G_{i_j}$, for all i_j .

- (2) For any $g \in G_1 * \dots * G_n$, the map $\rho(g) \in \text{Aut}(F_N)$ satisfies $\rho(g)(f) = \Delta \cdot f$, where $\Delta \in \Gamma^{k+1}(G_1 * \dots * G_n)$. That is, Δ is trivial in $\text{gr}_p(G_1 * \dots * G_n)$ for $p \leq k$.

In the last part of the paper we investigate the possible implications of the representation

$$\rho_{K_0} : \prod_{i=1}^n G_i \rightarrow \text{Out}(F_{N_n}),$$

corresponding to the zero simplicial complex with n vertices, on the representations

$$\rho_K : \prod_{i=1}^n G_i \rightarrow \text{Out}(\pi_1(Z_K(\underline{EG}, \underline{G}))),$$

where K is an arbitrary simplicial complex on n vertices. The motivation comes from the fact that there is a homotopy equivalence $Z_K(\underline{EG}, \underline{G}) \simeq Z_K([0, 1], \underline{F})$. We are able to reduce the question to the existence of certain commutative diagrams. More precisely the question is reduced to the existence of a map $r : \text{Out}(F_{N_n}) \rightarrow \text{Out}(\pi)$, where $\pi = \pi_1(Z_K(\underline{EG}, \underline{G}))$, such that the following diagram commutes

$$\begin{array}{ccc} & & \text{Out}(F_{N_n}) \\ & \nearrow \rho_{K_0} & \vdots r \\ G_1 \times \dots \times G_n & & \text{Out}(\pi) \\ & \searrow \rho_K & \end{array}$$

2. POLYHEDRAL PRODUCTS

In this section we give a short introduction to polyhedral products. Let $(\underline{X}, \underline{A})$ denote a finite sequence of pointed CW -pairs $\{(X_i, A_i)\}_{i=1}^n$. Let $[n]$ denote the set of natural numbers $\{1, 2, \dots, n\}$.

Definition 2.1. A *simplicial complex* K on the set $[n]$ is a subset of the power set of $[n]$, such that, if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.

A simplex $\sigma \in K$ is given by an increasing sequence of integers

$$\sigma = \{i_1, i_2, \dots, i_q\},$$

where $1 \leq i_1 < i_2 < \dots < i_q \leq n$. In particular, the empty set \emptyset is an element of K . The *geometric realization* $|K|$ of K is a simplicial complex inside the simplex $\Delta[n-1]$.

Define a functor

$$D : K \longrightarrow CW_*,$$

where CW_* denotes the category of pointed CW -complexes. For any $\sigma \in K$ let

$$D(\sigma) = \prod_{i=1}^n Y_i = Y_1 \times \cdots \times Y_n \subseteq \prod_{i=1}^n X_i,$$

where

$$Y_i = \begin{cases} A_i & : i \notin \sigma, \\ X_i & : i \in \sigma. \end{cases}$$

Definition 2.2. The *polyhedral product* $Z_K(\underline{X}, \underline{A})$ is the subset of the product $X_1 \times \cdots \times X_n$ given by the colimit

$$Z_K(\underline{X}, \underline{A}) = \operatorname{colim}_{\sigma \in K} D(\sigma) = \bigcup_{\sigma \in K} D(\sigma) \subseteq \prod_{i=1}^n X_i,$$

where the maps are the inclusions and the topology is the subspace topology.

In other words the polyhedral product is the colimit of the diagram of spaces $D(\sigma)$. The following notations appear throughout the literature and all represent the same polyhedral product: $Z_K(\underline{X}, \underline{A})$, $Z_K(X_i, A_i)$, $Z(K; (\underline{X}, \underline{A}))$ and $(\underline{X}, \underline{A})^K$. If the sequence of pairs is constant then we simply write $Z_K(X, A)$.

Example 2.3. Assume K is the zero simplicial complex $\{\{1\}, \dots, \{n\}\}$, $X_i = X$ and A_i be the basepoint $*$ in X . Then

$$Z_K(\underline{X}, \underline{A}) = Z_K(X, *) = X \vee \cdots \vee X,$$

the n -fold wedge sum of the space X . On the other hand, if K is the full simplex, then

$$Z_K(X, *) = X_1 \times \cdots \times X_n.$$

Example 2.4. Assume $K = \{\{1\}, \{2\}\}$. Let $(\underline{X}, \underline{A}) = (D^n, S^{n-1})$, the pair consisting of an n -disk and the bounding $(n-1)$ -sphere. Then

$$Z_K(\underline{X}, \underline{A}) = Z_K(D^n, S^{n-1}) = D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial D^{2n} = S^{2n-1}.$$

Definition 2.5. Given a simplicial graph Γ with vertex set S and a family of groups $\{G_s\}_{s \in S}$, their *graph product*

$$\prod_{\Gamma} G_s$$

is the quotient of the free product of the groups G_s by the relations that elements of G_s and G_t commute whenever $\{s, t\}$ is an edge of Γ . Note that if Γ is the complete graph on n vertices, then $\prod_{\Gamma} G_s \cong \prod_{i=1}^n G_i$.

Definition 2.6. K is a *flag complex* if any finite set of vertices, which are pairwise connected by edges, spans a simplex in K .

3. MONODROMY REPRESENTATION

Let G_1, \dots, G_n be finite discrete groups of order $|G_i| = m_i$, for $1 \leq i \leq n$. In this section we are interested in describing the monodromy representation corresponding to the Denham–Suciu fibration (equation 1.2) given by

$$Z_K(\underline{EG}, \underline{G}) \longrightarrow Z_K(\underline{BG}) \longrightarrow \prod_{i=1}^n BG_i.$$

Recall that the homotopy type of the polyhedral product $Z_K(\underline{X}, \underline{A})$ depends only on the relative homotopy type of the pairs $(\underline{X}, \underline{A})$.

Lemma 3.1. *Let G be a finite discrete group of order m . Then there is a relative homotopy equivalence $(EG, G) \sim ([0, 1], F)$, where F is a subset of $[0, 1]$ of cardinality m .*

Proof. See [6, Lemma 3.5]. □

Hence, there is a homotopy equivalence $Z_K(\underline{EG}, \underline{G}) \simeq Z_K(\underline{I}, \underline{F})$, where I is the unit interval $[0, 1]$ and $(\underline{I}, \underline{F}) = \{(I, F_j)\}_{j=1}^n$. If $K = K_0$ is the zero skeleton of the $(n-1)$ -simplex, then $Z_{K_0}(\underline{I}, \underline{F})$ is a connected simplicial graph embedded in the space $[0, 1]^n \subset \mathbb{R}^n$ (see Figure 1) and hence, has the homotopy type of a wedge of N_n circles. As mentioned in the introduction and in [6, Proposition 3.6], the integer N_n is given by

$$N_n = (n-1) \prod_{i=1}^n - \sum_{i=1}^n \left(\prod_{j \neq i} m_j \right) + 1.$$

Lemma 3.1 allows for a geometric description of the fibre in the Denham–Suciu fibration for any K , and for the case of the zero simplicial complex in particular. This is a geometric model that will sometimes be used to describe monodromy concretely. This description of the fibre depends only on the order of the groups G_i , but clearly the monodromy representations still depend on the structure of the groups. For some computations we will restrict to finite cyclic groups, but we will also describe the method for other finite discrete groups.

3.1. Generators for the fundamental group.

Let K_0 denote the 0-simplicial complex on n vertices. In this section we describe explicit loops in $Z_{K_0}(\underline{I}, \underline{F})$, whose equivalence classes constitute a generating set for the fundamental group, F_{N_n} . Recall that the simplicial complex K_0 is a flag complex, thus the spaces in the Denham–Suciu fibration are Eilenberg–Mac Lane spaces and there is a short exact sequence of groups

$$1 \longrightarrow F_{N_n} \longrightarrow G_1 * \dots * G_n \longrightarrow G_1 \times \dots \times G_n \longrightarrow 1,$$

where $G_1 * \dots * G_n$ denotes the free product of the groups. The classes of loops that we will find are therefore generators for the free group F_{N_n} in the short exact sequence.

The homotopy type of $Z_K(\underline{EG}, \underline{G})$ depends only on the cardinality of G_i . Hence, when finding the loops for $Z_K(\underline{EG}, \underline{G})$ for finite cyclic groups G_1, \dots, G_n , the same computation holds for any collection of groups with the same order, that is the

same classes of loops will be used to describe the monodromy. However, the representation depends on the structure of the groups.

Definition 3.2. The loops are described as follows: Let $*$ $= (0, \dots, 0) \in I^n$ be the basepoint of $Z_{K_0}(\underline{I}, \underline{F})$, which is the image of $(1_{G_1}, \dots, 1_{G_n})$ under the homotopy equivalence. Starting from the basepoint $*$, each path in $Z_{K_0}(\underline{I}, \underline{F})$ will be tracked by a word $\omega = x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_r}^{j_r}$, where $x_{i_k}^{j_k} \in G_k$, each letter x_k showing the coordinate of the group it belongs to, together with the exponent j_k showing the distance taken in that direction. See Figure 1 for a picture in two dimensions. For any word $\omega \in F_{N_n}$, let γ_ω denote the path in $Z_{K_0}(\underline{I}, \underline{F})$ tracked by the word ω .

Lemma 3.3. *The path tracked by the word $x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_r}^{j_r}$ is closed if and only if $\sum_{i=1}^r j_i = 0$.*

Proof. This can be seen by arguing that, to start and end at the basepoint $*$, if $x_{i_1}^{j_1}$ is a letter of the word, then the letter $x_{i_1}^{-j_1}$ should also appear in the same word, otherwise one can never come back to $*$. Conversely if the sum $\sum_{i=1}^r j_i = 0$, then every move forward has been compensated by a move backward. \square

3.2. Generators for the case of two finite groups.

Let G_1 and G_2 be finite cyclic groups with order m and n , respectively, such that $G_1 = \{1, x_1, x_1^2, \dots, x_1^{m-1}\}$ and $G_2 = \{1, x_2, x_2^2, \dots, x_2^{n-1}\}$. The zero simplicial complex with two vertices is $K_0 = \{\{1\}, \{2\}\}$. Assume there are bijections of finite sets $G_1 \approx F_1$ and $G_2 \approx F_2$ given by

$$G_1 = \{1, x_1, x_1^2, \dots, x_1^{m-1}\} \approx F_1 = \{0 = t_{1,0} < t_{1,1} < \dots < t_{1,m-1} = 1\} \subset I,$$

$$G_2 = \{1, x_2, x_2^2, \dots, x_2^{n-1}\} \approx F_2 = \{0 = t_{2,0} < t_{2,1} < \dots < t_{2,n-1} = 1\} \subset I,$$

identifying x_i^k with $t_{i,k}$. Then from Definition 2.2 we have the following

$$Z_{K_0}(\underline{EG}, \underline{G}) \simeq Z_{K_0}(\underline{I}, \underline{F}) = D(\{1\}) \cup D(\{2\}) = I \times F_2 \cup F_1 \times I,$$

see Figure 1. Consider the cycles γ_ω described in Definition 3.2 starting at the basepoint $*$ $= (0, 0)$, given by the words

$$\omega = [x_1^i, x_2^j] = x_1^i x_2^j x_1^{-i} x_2^{-j},$$

where $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. The following lemma tells which loops suffice.

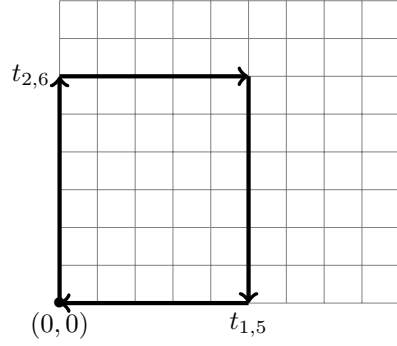
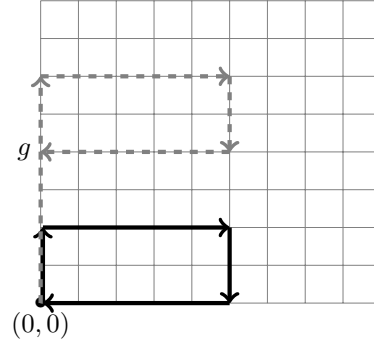
Lemma 3.4. *The set of words $\mathcal{W} = \{[x_1^i, x_2^j] | 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$ generates all the cycles $\gamma_\omega \in Z_{K_0}(\underline{I}, \underline{F})$.*

Proof. Take an arbitrary word of finite length

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} x^{m_3} y^{n_3} \dots x^{m_k} y^{n_k} x^{m_{k+1}}.$$

Then it can be written as a product of commutators as follows:

$$\begin{aligned} & x^{m_1} y^{n_1} x^{m_2} y^{n_2} x^{m_3} y^{n_3} \dots x^{m_k} y^{n_k} x^{m_{k+1}} \\ &= [x^{m_1}, y^{n_1}] \cdot [y^{n_1}, x^{m_1+m_2}] \cdot [x^{m_1+m_2}, y^{n_1+n_2}] \cdot [y^{n_1+n_2}, x^{m_1+m_2+m_3}] \\ & \quad \cdot [x^{m_1+m_2+m_3}, y^{n_1+n_2+n_3}] \dots [y^{n_1+\dots+n_k}, x^{m_1+\dots+m_k+m_{k+1}}]. \end{aligned}$$

FIGURE 1. 2-dim. case,
the loop $[x_1^6, x_2^5]$ FIGURE 2. $g = x_2^4$ act-
ing on $[x_1^2, x_2^5]$

Since any cycle can be described by such a word, this suffices. \square

Lemma 3.5. *The set of words $\mathcal{W} = \{[x_1^i, x_2^j] | 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$ is a minimal generating set.*

Proof. First note that $|\mathcal{W}| = (m-1)(n-1) = mn - (m+n) + 1$. Then it follows from [6, Proposition 3.6] that $N_2 = |\mathcal{W}|$. \square

Now let H_1 and H_2 be finite discrete groups with cardinality m and n , respectively. That is, $H_1 = \{1, g_1, \dots, g_{m-1}\}$ and $H_2 = \{1, h_1, \dots, h_{n-1}\}$.

Corollary 3.6. *The set of words*

$$\mathcal{W}' = \{[g_i, h_j] | 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$$

generates all the cycles in $Z_{K_0}(\underline{EH}, \underline{H}) \simeq Z_{K_0}(\underline{I}, \underline{F})$. Moreover, this is a minimal generating set.

Proof. Take an arbitrary word of finite length

$$g_{m_1} h_{n_1} g_{m_2} h_{n_2} g_{m_3} h_{n_3} \cdots g_{m_k} h_{n_k} g_{m_{k+1}}.$$

Then it can be written as a product of commutators as follows:

$$\begin{aligned} & g_{m_1} h_{n_1} g_{m_2} h_{n_2} g_{m_3} h_{n_3} \cdots g_{m_k} h_{n_k} g_{m_{k+1}} \\ &= [g_{m_1}, h_{n_1}] \cdot [h_{n_1}, g_{m_1} g_{m_2}] \cdot [g_{m_1} g_{m_2}, h_{n_1} h_{n_2}] \cdot [h_{n_1} h_{n_2}, g_{m_1} g_{m_2} g_{m_3}] \\ & \quad \cdot [g_{m_1} g_{m_2} g_{m_3}, h_{n_1} h_{n_2} h_{n_3}] \cdots [h_{n_1} \cdots h_{n_k}, g_{m_1} \cdots g_{m_{k+1}}]. \end{aligned}$$

Since any cycle can be described by such a word, this suffices. Now we have $|\mathcal{W}'| = (m-1)(n-1) = mn - (m+n) + 1$. Then it follows from [6, Proposition 3.6] that $N_2 = |\mathcal{W}'|$, thus giving minimality. \square

The next step is to describe the action $G_1 \times \cdots \times G_n$ on these generators. We know $G_1 \times \cdots \times G_n$ acts on the loops in the fiber by conjugation, that is,

$$g \cdot \gamma_\omega = \gamma_{g\omega g^{-1}},$$

If we refer to Figure 2, this action shifts the loop by g . For example, let $G_1 = C_{10}$ and $G_2 = C_9$ be the cyclic groups of order 10 and 9, respectively. The element $x_2^4 \in G_2$ acts on the word $\omega = [x_1^2, x_2^5]$ by conjugation

$$x_2^4 \cdot \omega = x_2^4 \cdot [x_1^2, x_2^5] = x_2^4 \omega x_2^{-4}.$$

Therefore

$$x_2^4 \cdot \gamma_\omega = \gamma_{x_2^4 \omega x_2^{-4}},$$

which is the loop γ_ω shifted by $x_2^4 \in G_2$ in the direction of G_2 .

3.3. The monodromy action.

Assume G_1, \dots, G_n are finite discrete groups. Let \mathcal{W} be a minimal set of generators for $\pi_1(Z_{K_0}(\underline{I}, \underline{F})) \cong F_{N_n}$. Let $[\gamma_\omega] = w \in F_{N_n}$ be the homotopy class of the loop γ_ω in $Z_{K_0}(\underline{I}, \underline{F})$, where $\omega \in \mathcal{W}$. Then $G_1 \times \dots \times G_n$ acts on the fundamental group of the fibre $Z_{K_0}(\underline{I}, \underline{F})$ as follows

$$g \cdot \gamma_\omega = \gamma_{g\omega g^{-1}},$$

that is,

$$g \cdot \omega = g \cdot [\gamma_\omega] = [\gamma_{g\omega g^{-1}}] = g\omega g^{-1}.$$

The goal here is to write $g\omega g^{-1}$ as a product of words in \mathcal{W} . Then any element g in the free product $G_1 * \dots * G_n$ gives an automorphism of F_{N_n} , the free group with generators the elements of \mathcal{W}

$$\begin{aligned} G_1 * \dots * G_n &\xrightarrow{\varphi} \text{Aut}(F_{N_n}) \\ g &\longmapsto \varphi_g, \end{aligned}$$

where $\text{Aut}(G)$ is the group of group automorphisms of G , under composition. One example is given in Section 4.1.

In general, recall that given a short exact sequence of discrete groups

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1,$$

with A a normal subgroup of B , there is a map

$$\begin{aligned} B &\xrightarrow{\Theta} \text{Aut}(A) \\ g &\longmapsto \Theta(g) \end{aligned}$$

such that $\Theta(g)(h) = ghg^{-1}$. There is also a map

$$\begin{aligned} B &\xrightarrow{\Psi} \text{Inn}(A) \\ g &\longmapsto \Psi(g) \end{aligned}$$

such that $\Psi(g)(h) = ghg^{-1}$, where $\text{Inn}(A)$ is the group of *inner automorphisms* of A . Moreover, $\text{Inn}(A) \trianglelefteq \text{Aut}(A)$ and $\text{Out}(A) := \text{Aut}(A)/\text{Inn}(A)$ is the group of *outer automorphisms* of A . Note that for $A = F_n$ a free group, $F_n \cong \text{Inn}(F_n)$.

For the free group F_n and $n \geq 2$, there is a short exact sequence of groups

$$1 \longrightarrow \text{Inn}(F_n) \longrightarrow \text{Aut}(F_n) \longrightarrow \text{Out}(F_n) \longrightarrow 1$$

and hence, a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & F_{N_n} & \longrightarrow & G_1 * \cdots * G_n & \longrightarrow & G_1 \times \cdots \times G_n \longrightarrow 1 \\
& & \downarrow \Psi & & \downarrow \Theta & & \downarrow \tilde{\Theta} \\
1 & \longrightarrow & \text{Inn}(F_{N_n}) & \longrightarrow & \text{Aut}(F_{N_n}) & \longrightarrow & \text{Out}(F_{N_n}) \longrightarrow 1,
\end{array}$$

where G_1, \dots, G_n are finite discrete groups. So the map

$$\Theta : G_1 * \cdots * G_n \rightarrow \text{Aut}(F_{N_n})$$

induces a map

$$\tilde{\Theta} : G_1 \times \cdots \times G_n \rightarrow \text{Out}(F_{N_n}),$$

which is the representation we are interested in.

There is also another short exact sequence

$$1 \longrightarrow \text{IA}_n \longrightarrow \text{Aut}(F_n) \xrightarrow{\text{ab}} \text{GL}_n(\mathbb{Z}) \longrightarrow 1,$$

with kernel the group IA_n , which is the subgroup of automorphisms that restrict to the identity in the abelianization of F_n , and “ab” is the map induced by the abelianization map $F_n \rightarrow F_n/[F_n, F_n] = \mathbb{Z}^n$. In the examples that will be given, none of the homomorphisms restrict to the identity in the abelianization. Thus, these elements are not elements of IA_n . Equivalently, this says that the fundamental group of the base acts non-trivially on the homology of the fibre.

4. EXAMPLES

Example 4.1. Consider the groups

$$G_1 = \mathbb{Z}/2\mathbb{Z} := \mathbb{Z}_2 = \langle x_1 | x_1^2 = 1 \rangle \text{ and } G_2 = \mathbb{Z}/3\mathbb{Z} := \mathbb{Z}_3 = \langle x_2 | x_2^3 = 1 \rangle.$$

Then using the Denham–Suciu fibration there is the following short exact sequence of groups

$$1 \longrightarrow F_2 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow 1,$$

where F_2 is the free group on the generators $\omega_1 = [x_1, x_2]$ and $\omega_2 = [x_1, x_2^2]$.

To compute the map $\Theta : \mathbb{Z}_2 * \mathbb{Z}_3 \rightarrow \text{Aut}(F_2)$, we first compute the automorphism $\varphi_{x_1} \in \text{Aut}(F_2)$ by looking at the image of the generators $\omega_1, \omega_2 \in F_2$ under φ_{x_1} to find

$$x_1 \omega_1 x_1^{-1} = [x_2, x_1] = ([x_1, x_2])^{-1} = \omega_1^{-1}$$

and

$$x_1 \omega_2 x_1^{-1} = [x_2^2, x_1] = ([x_1, x_2^2])^{-1} = \omega_2^{-1}.$$

Looking at the induced map of φ_{x_1} onto the abelianization $\mathbb{Z} \oplus \mathbb{Z} \cong F_2/[F_2, F_2]$, then

$$\tilde{\varphi}_{x_1}(\omega_1, \omega_2) = (-\omega_1, -\omega_2),$$

which is given by the matrix

$$[\tilde{\varphi}_{x_1}] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the basis $\{\omega_1, \omega_2\}$. Then the other representations can be given by $[\tilde{\varphi}_{x_1^i}] = [\tilde{\varphi}_{x_1}]^i$. Similarly, one can compute $\varphi_{x_2} \in \text{Aut}(F_2)$ by finding

$$x_2 \omega_1 x_2^{-1} = x_2 [x_1, x_2] x_2^{-1} = [x_2, x_1] [x_1, x_2^2] = \omega_1^{-1} \omega_2$$

and

$$x_2 \omega_2 x_2^{-1} = [x_2, x_1] = ([x_1, x_2])^{-1} = \omega_1^{-1}.$$

Looking at the induced map of φ_{x_2} onto the abelianization $\mathbb{Z} \oplus \mathbb{Z} \cong F_2/[F_2, F_2]$, then

$$\tilde{\varphi}_{x_2}(\omega_1, \omega_2) = (-\omega_1 + \omega_2, -\omega_1),$$

which is given by the matrix

$$[\tilde{\varphi}_{x_2}] = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the basis $\{\omega_1, \omega_2\}$. Similarly, $[\tilde{\varphi}_{x_2^i}] = [\tilde{\varphi}_{x_2}]^i$. Using properties of group actions, any automorphism $\varphi_g, g \in F_2$ can be found using φ_{x_1} and φ_{x_2} . For example $\varphi_{x_1 x_2} = \varphi_{x_1} \circ \varphi_{x_2}$ and so on. Note that φ_{x_1} and φ_{x_2} are not elements of IA_2 since the functions do not restrict to the identity in the abelianization. Hence, the fundamental group of the base $\mathbb{Z}_2 \times \mathbb{Z}_3$ acts non-trivially on the homology of the fibre, as mentioned in the previous section.

This calculation gives a homomorphism $\text{ab} \circ \Theta : \mathbb{Z}_2 * \mathbb{Z}_3 \rightarrow GL_2(\mathbb{Z})$ by composing the homomorphisms

$$\mathbb{Z}_2 * \mathbb{Z}_3 \xrightarrow{\Theta} \text{Aut}(F_2) \xrightarrow{\text{ab}} GL_2(\mathbb{Z}).$$

The map Θ induces a homomorphism $\tilde{\Theta} : \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow \text{Out}(F_2)$. Moreover, the map $\text{ab} \circ \Theta$ can be considered the same as the composition $p \circ \tilde{\Theta}$, where p is the projection to the abelianization of $\mathbb{Z}_2 * \mathbb{Z}_3$, since $[\tilde{\varphi}_{x_1}]$ and $[\tilde{\varphi}_{x_2}]$ commute.

Example 4.2. Let Σ_3 be the symmetric group on three letters, given by

$$\Sigma_3 = \{1, (12), (13), (23), (123), (132)\}.$$

Let $C_2 = \mathbb{Z}_2 = \{1, x\}$ be the cyclic group with two elements. There is a short exact sequence of groups

$$1 \longrightarrow F_5 \longrightarrow \mathbb{Z}_2 * \Sigma_3 \longrightarrow \mathbb{Z}_2 \times \Sigma_3 \longrightarrow 1,$$

where F_5 is the free group on letters $W = \{[x, g] | x, g \neq 1, x \in \mathbb{Z}_2, g \in \Sigma_3\}$. To calculate the representation $\mathbb{Z}_2 \times \Sigma_3 \rightarrow \text{Out}(F_5)$, start with evaluating φ_x for $x \in \mathbb{Z}_2$. Hence, $\varphi_x([x, g]) = [g, x] = [x, g]^{-1}$ for all $g \in \Sigma_3$. After restricting to the abelianization $\tilde{\varphi}_x([x, g]) = -[x, g]$. Hence, the matrix representation of $\tilde{\varphi}_x$ is given by $[\tilde{\varphi}_x] = -I_5$.

Similarly, $\varphi_{(12)}([x, (12)]) = [(12), x]$ and $\varphi_{(12)}([x, g]) = [(12), x][x, (12) \cdot g]$ if $g \neq (12)$. In the abelianization, we get $\tilde{\varphi}_{(12)}([x, (12)]) = -[x, (12)]$ and $\tilde{\varphi}_{(12)}([x, g]) = -[x, (12)] + [x, (12)g]$. Order the basis as follows

$$W = \{[x, (12)], [x, (13)], [x, (23)], [x, (123)], [x, (132)]\}.$$

Then the matrix representation for $\tilde{\varphi}_{(12)}$ is

$$[\tilde{\varphi}_{(12)}] = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

One can find the other automorphisms similarly, since $\varphi_g([x, h]) = [g, x][x, gh]$. Hence,

$$[\tilde{\varphi}_{(13)}] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, [\tilde{\varphi}_{(23)}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[\tilde{\varphi}_{(123)}] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, [\tilde{\varphi}_{(132)}] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

Note that these matrices do not commute in general. For example $[\tilde{\varphi}_{(13)}] \cdot [\tilde{\varphi}_{(132)}] \neq [\tilde{\varphi}_{(132)}] \cdot [\tilde{\varphi}_{(13)}]$. However, $[\tilde{\varphi}_x]$ commutes with the other matrices. Hence, the map

$$\mathbb{Z}_2 * \Sigma_3 \xrightarrow{\Theta} \text{Aut}(F_5) \xrightarrow{\text{ab}} GL_5(\mathbb{Z})$$

is the same as the composition

$$\mathbb{Z}_2 * \Sigma_3 \xrightarrow{p} \mathbb{Z}_2 \times \Sigma_3 \xrightarrow{\tilde{\Theta}} \text{Out}(F_5) \xrightarrow{\text{ab}} GL_5(\mathbb{Z}).$$

Therefore, there is a homomorphism $\mathbb{Z}_2 \times \Sigma_3 \rightarrow GL_5(\mathbb{Z})$. Also note that

$$\det([\tilde{\varphi}_{(12)}]) = \det([\tilde{\varphi}_{(13)}]) = \det([\tilde{\varphi}_{(23)}]) = -1,$$

and consequently

$$\det([\tilde{\varphi}_{(123)}]) = \det([\tilde{\varphi}_{(132)}]) = 1.$$

4.1. Two finite cyclic groups.

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. Consider the general case of two cyclic groups

$$G_1 \cong \mathbb{Z}/r\mathbb{Z} \cong \langle x_1 | x_1^r = 1 \rangle \text{ and } G_2 \cong \mathbb{Z}/m\mathbb{Z} \cong \langle x_2 | x_2^m = 1 \rangle.$$

There is a short exact sequence of groups

$$1 \longrightarrow F_k \longrightarrow \mathbb{Z}_r * \mathbb{Z}_m \longrightarrow \mathbb{Z}_r \times \mathbb{Z}_m \longrightarrow 1,$$

coming from the Denham–Suciu fibration, where F_k is the free group on $k = (r-1)(m-1)$ letters given by the elements of

$$\mathcal{W}_2 = \{\omega_{ij} = [x_1^i, x_2^j] | 1 \leq i \leq r-1, 1 \leq j \leq m-1\}.$$

To compute the map $\Theta : \mathbb{Z}_r * \mathbb{Z}_m \rightarrow \text{Aut}(F_k)$, we first compute the automorphism $\varphi_{x_1} \in \text{Aut}(F_k)$ by looking at the image of the generators $\omega_{ij} \in F_k$ under φ_{x_1} to find

$$x_1 \omega_{ij} x_1^{-1} = x_1 [x_1^i, x_2^j] x_1^{-1} = [x_1^{i+1}, x_2^j] [x_2^j, x_1] = \omega_{i+1,j} \omega_{1,j}^{-1}.$$

Looking at the induced map of φ_{x_1} onto the abelianization

$$\bigoplus_{(r-1)(m-1)} \mathbb{Z} \cong F_{(r-1)(m-1)} / [F_{(r-1)(m-1)}, F_{(r-1)(m-1)}]$$

then

$$\tilde{\varphi}_{x_1}(\omega_{11}, \dots, \omega_{(r-1)(m-1)}) = (\omega_{2,1} - \omega_{1,1}, \omega_{2,2} - \omega_{1,2}, \dots, -\omega_{(r-1),(m-1)})$$

which is given by the matrix

$$(4.1) \quad [\tilde{\varphi}_{x_1}] = \begin{pmatrix} -I_{m-1} & I_{m-1} & 0 & 0 & \cdots & 0 \\ 0 & -I_{m-1} & I_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & -I_{m-1} & I_{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -I_{m-1} & I_{m-1} \\ 0 & \cdots & 0 & 0 & 0 \cdots & -I_{m-1} \end{pmatrix}$$

with respect to the basis \mathcal{W}_2 , where I_{m-1} is the $(m-1) \times (m-1)$ identity matrix. Hence, clearly φ_{x_1} is not an element of IA_k .

For $\varphi_{x_2} \in \text{Aut}(F_k)$:

$$x_2 \omega_{ij} x_2^{-1} = x_2 [x_1^i, x_2^j] x_2^{-1} = [x_2, x_1^i] [x_1^i, x_2^{j+1}] = \omega_{i,1}^{-1} \omega_{i,j+1}.$$

Similarly, looking at the induced map of φ_{x_2} onto the abelianization of F_k we get

$$\tilde{\varphi}_{x_2}(\omega_{11}, \dots, \omega_{(r-1)(m-1)}) = (-\omega_{1,1} + \omega_{1,2} - \omega_{1,1} + \omega_{1,3}, \dots, -\omega_{(r-1),(m-1)}),$$

which is given by the matrix

$$(4.2) \quad [\tilde{\varphi}_{x_2}] = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{r-1} \end{pmatrix}$$

with respect to the basis \mathcal{W}_2 , where

$$A_i = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(m-1) \times (m-1)}$$

for all i . Hence, φ_{x_2} is not an element of IA_k .

In general, Θ maps an element $x_1^i x_2^j$ to $\varphi_{x_i \cdot x_j} \in \text{Aut}(F_k)$, which when restricted to the abelianization $\bigoplus_k \mathbb{Z}$, can be identified with the matrix $[\tilde{\varphi}_{x_1}]^i [\tilde{\varphi}_{x_2}]^j$. This matrix is the identity if and only if $i = r$ and $j = m$. Hence, there is a homomorphism

$$\mathbb{Z}_r * \mathbb{Z}_m \xrightarrow{\text{abo}\Theta} GL_k(\mathbb{Z}).$$

Θ induces a homomorphism $\tilde{\Theta} : \mathbb{Z}_r \times \mathbb{Z}_m \longrightarrow \text{Out}(F_k)$. Hence, there is a homomorphism $\mathbb{Z}_r \times \mathbb{Z}_m \xrightarrow{\text{abo}\tilde{\Theta}} GL_k(\mathbb{Z})$.

If m is even and r is odd or vice versa, then

$$\det[\tilde{\varphi}_{x_1}] = (-1)^{(r-1)(m-1)} = 1,$$

$$\det[\tilde{\varphi}_{x_2}] = \det(A_1) \cdots \det(A_{r-1}) = (\det(A_1))^{r-1}.$$

Since $\det(A_i) = 1$ if m is odd and -1 if m is even, and if r is odd we get $(-1)^{r-1} = 1$, then $\det[\tilde{\varphi}_{x_2}] = 1$. Hence, there is a homomorphism

$$\mathbb{Z}_r * \mathbb{Z}_m \longrightarrow SL_k(\mathbb{Z})$$

which induces a homomorphism

$$\begin{array}{ccc} \mathbb{Z}_r * \mathbb{Z}_m & \xrightarrow{\text{ab} \circ \tilde{\Theta} \circ \text{ab}} & SL_k(\mathbb{Z}) \subset GL_k(\mathbb{Z}) \\ & \searrow \text{ab} & \nearrow \text{ab} \\ & \mathbb{Z}_r \times \mathbb{Z}_m \xrightarrow{\tilde{\Theta}} \text{Out}(F_k) & \end{array}$$

That is, there is a representation of $\mathbb{Z}_r \times \mathbb{Z}_m \rightarrow SL_k(\mathbb{Z})$. Similarly as before, the map $\text{ab} \circ \tilde{\Theta}$ can be considered the same as the composition $p \circ \tilde{\Theta}$, where p is the projection to the abelianization of $\mathbb{Z}_r * \mathbb{Z}_m$, since $[\tilde{\varphi}_{x_1}]$ and $[\tilde{\varphi}_{x_2}]$ commute. To show that $[\tilde{\varphi}_{x_1}]$ and $[\tilde{\varphi}_{x_2}]$ commute it suffices to show that they commute for $r = m = 3$

$$[\tilde{\varphi}_{x_1}] \cdot [\tilde{\varphi}_{x_2}] = [\tilde{\varphi}_{x_2}] \cdot [\tilde{\varphi}_{x_1}] = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

□

4.2. Two arbitrary finite groups.

We begin by proving Proposition 1.1.

Proof of Proposition 1.1. Let G and H be finite discrete groups, not necessarily cyclic or abelian, with cardinality m and n respectively. That is, assume

$$G = \{1, g_1, \dots, g_{m-1}\} \text{ and } H = \{1, h_1, \dots, h_{n-1}\}.$$

There is a short exact sequence of groups

$$1 \longrightarrow F_{(m-1)(n-1)} \longrightarrow G * H \longrightarrow G \times H \longrightarrow 1$$

coming from the Denham–Suciú fibration, where the rank of the free group is determined by the formula in equation 1.3. To calculate the map

$$G * H \rightarrow \text{Aut}(F_{(m-1)(n-1)}),$$

start with φ_f , where $f \in G$ or $f \in H$. Choose a basis for $F_{(m-1)(n-1)}$ to be

$$W = \{[g_i, h_j] \mid 1 \leq i \leq m-1, 1 \leq j \leq n-1\}.$$

Then,

$$(4.3) \quad \begin{aligned} \varphi_{g_k}([g_i, h_j]) &= g_k [g_i, h_j] g_k^{-1} = [g_k g_i, h_j] [h_j, g_k] \\ \varphi_{h_k}([g_i, h_j]) &= h_k [g_i, h_j] h_k^{-1} = [h_k, g_i] [g_i, h_k h_j]. \end{aligned}$$

Note that the images $\varphi_{g_k}([g_i, h_j])$ and $\varphi_{h_k}([g_i, h_j])$ are trivial if and only if $g_k = 1$ and $h_k = 1$, respectively. Therefore, the representation is faithful. □

To find the matrix representation of these, it is necessary to know the group structure of G and H . Hence, we get a composition of homomorphisms

$$G * H \rightarrow G \times H \rightarrow \text{Out}(F_{(m-1)(n-1)})$$

which is the same as the composition

$$G * H \rightarrow \text{Aut}(F_{(m-1)(n-1)}) \rightarrow \text{Out}(F_{(m-1)(n-1)}).$$

Remark. In sections 4.1 and 4.2 as well as in the examples, data is being collected, with the goal of axiomatizing properties of the monodromy.

Proposition 4.3. *Let G and H be two finite discrete groups with cardinality m and n , respectively. Then there is a faithful representation*

$$G \times H \rightarrow \text{Out}(F_k),$$

given by equation 4.3, where $k = (m-1)(n-1)$.

4.3. A collection of finite discrete groups.

Recall that for a group G , there is a sequence of subgroups called the *descending central series* of G given by

$$G = \Gamma^1(G) \supseteq \Gamma^2(G) \supseteq \cdots \supseteq \Gamma^n(G) \supseteq \cdots$$

such that the second stage is $\Gamma^2(G) = [G, G]$ and the $(n+1)$ -st stage is given inductively by $\Gamma^{n+1}(G) = [\Gamma^n(G), G]$. The Lie algebra of G associated to the descending central series is given by

$$\text{gr}_*(G) = \bigoplus_{i \geq 1} \Gamma^i(G) / \Gamma^{i+1}(G)$$

with $\text{gr}_p(G) = \Gamma^p(G) / \Gamma^{p+1}(G)$.

Lemma 4.4. *Let $\{G_i\}_{i=1}^n$ be a collection of finite discrete groups and K_0 be the 0-simplicial complex on n vertices. Let $\rho : \prod_{i=1}^n G_i \rightarrow \text{Out}(F_N)$ be the monodromy representation where F_N is isomorphic to the kernel of the projection $p : G_1 * \cdots * G_n \rightarrow \prod_{i=1}^n G_i$. Then the following hold:*

- (1) *There is a choice of a generating set for F_N that consists of elements of the form*

$$f = [g_{i_1}, [g_{i_2}, [\cdots, [g_{i_{k-1}}, g_{i_k}] \cdots]]] \in \Gamma^k(G_1 * \cdots * G_n)$$

such that $g_{i_j} \in G_{i_j}$, for all i_j .

- (2) *For any $g \in G_1 * \cdots * G_n$, the map $\rho(g) \in \text{Aut}(F_N)$ satisfies $\rho(g)(f) = \Delta \cdot f$, where $\Delta \in \Gamma^{k+1}(G_1 * \cdots * G_n)$. That is, Δ is trivial in $\text{gr}_p(G_1 * \cdots * G_n)$ for $p \leq k$.*

Proof. Part 1: From the homotopy type of $Z_{K_0}(\underline{EG}, \underline{G}) \subset [0, 1]^n$ it is clear that all types of paths can be described using commutators of length at most n . It remains to prove that it is sufficient to consider only $g_{i_j} \in G_{i,j}$ and not other elements in $G_1 * \cdots * G_n$ to construct these commutators.

Start with $[g_i g_j, g_k] \in \Gamma^3(G_1 * \cdots * G_n)$. Then

$$[g_i g_j, g_k] = [g_i, [g_j, g_k]] \cdot [g_j, g_k][g_i, g_k].$$

Thus for any product, say $g_i = h_1 \cdots h_t$, it follows that

$$[g_i g_j, g_k] = [(h_1 \cdots h_t) g_j, g_k] = [h_1 \cdots h_t, [g_j, g_k]] \cdot [g_j, g_k] \cdot [h_1 \cdots h_t, g_k].$$

Then this product can be reduced to a product of commutators of the form stated in part 1, in finitely many steps by applying the step t more times.

Part 2: If $f = [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]] \in \Gamma^k(G_1 * \dots * G_n)$ is an element in F_N , then

$$\begin{aligned} \rho(g)(f) &= g \cdot [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]] \cdot g^{-1} \\ &= [g, [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]]] \cdot [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]] \\ &= \Delta \cdot f, \end{aligned}$$

where $\Delta = [g, [g_{i_1}, [g_{i_2}, [\dots, [g_{i_{k-1}}, g_{i_k}] \dots]]]] = [g, f] \in \Gamma^{k+1}(G_1 * \dots * G_n)$. \square

Finally, in the following remark we discuss the implications that these representations might have for the monodromy for any flag complex K .

Remark. Consider the Denham and Suciu fibration for flag complexes K and finite discrete groups G_1, \dots, G_n , and consider the corresponding monodromy representation

$$\rho_K : G_1 \times \dots \times G_n \longrightarrow \text{Out}(\pi_1(Z_K(\underline{EG}, \underline{G}))).$$

We are interested in a possible relation between ρ_K and

$$\rho_{K_0} : G_1 \times \dots \times G_n \longrightarrow \text{Out}(F_N),$$

where F_N is the kernel of the projection $G_1 * \dots * G_n \twoheadrightarrow G_1 \times \dots \times G_n$. We are lead to believe that solving ρ_{K_0} will help solve the other representations ρ_K because of the geometric description of monodromy. The action of the fundamental group of the base shifts loops of the fibre in a *certain direction*. On the other hand adding higher dimensional faces to K_0 will kill loops in the fibre in a way that can be described precisely (e.g. adding an edge kills loops *parallel* to each other etc.). Then the monodromy for the new K can be described, at least geometrically, using the monodromy for K_0 . As an illustration, figures 3 and 4 give the fibre of the Denham-Suciu fibration for the choice of $G_1 = G_2 = G_3 = \mathbb{Z}/2\mathbb{Z}$ with K_0 and K , respectively, where K_0 has only three vertices and K has an extra edge. Here ρ_{K_0} is supposed to determine ρ_K since the loops in the shaded faces are killed.

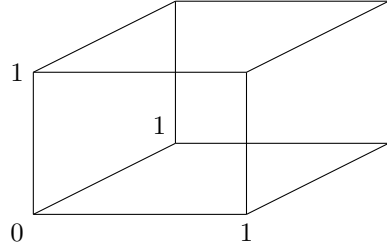


FIGURE 3. $G_i = \mathbb{Z}/2\mathbb{Z}$, K_0

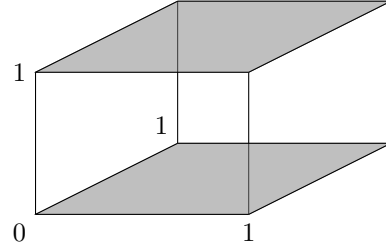


FIGURE 4. $G_i = \mathbb{Z}/2\mathbb{Z}$, K

A similar situation would occur in higher dimensions, where if the faces $\sigma_1, \dots, \sigma_s$ are added to K_0 to obtain K , the monodromy for K would be extracted from the monodromy for K_0 , by keeping track of the order in which the faces are added and which loops are killed. This is believed to work *a priori* since monodromy shifts non-trivial loops in the direction where the loops are not killed.

One way to attack this problem algebraically is as follows: there is a commutative diagram of fibrations

$$(4.4) \quad \begin{array}{ccccc} H(p) & \longrightarrow & H(p) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ Z_{K_0}(\underline{EG}, \underline{G}) & \longrightarrow & Z_{K_0}(\underline{BG}) & \longrightarrow & \prod_{i=1}^n BG_i \\ p \downarrow & & \downarrow & & \downarrow \\ Z_K(\underline{EG}, \underline{G}) & \longrightarrow & Z_K(\underline{BG}) & \longrightarrow & \prod_{i=1}^n BG_i, \end{array}$$

where $H(p)$ is the homotopy fibre of the map p . The fibre $H(p)$ is connected, so it follows from the long exact sequence in homotopy that the map p induces a surjection

$$p_{\#} : \pi_1(Z_{K_0}(\underline{EG}, \underline{G})) \rightarrow \pi_1(Z_K(\underline{EG}, \underline{G}))$$

on the level of fundamental groups. Thus, the kernel of the projection map is a free group, say F_q . From [6, Theorem 1.1] it follows that both the fibre $H(p)$ and $Z_K(\underline{EG}, \underline{G})$ are Eilenberg–Mac Lane spaces. Assume $Z_K(\underline{EG}, \underline{G})$ has fundamental group π .

Let F_N be the kernel of the projection $G_1 * \cdots * G_n \rightarrow \prod_{i=1}^n G_i$. Consider the commutative diagram of fibrations in (4.4). If K is a flag complex, then there is a commutative diagram as follows

$$(4.5) \quad \begin{array}{ccccccc} F_q & \xrightarrow{\cong} & F_q & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ F_N & \longrightarrow & G_1 * \cdots * G_n & \longrightarrow & \prod G_i \\ \downarrow p_{\#} & \searrow & \downarrow & \searrow \rho_{K_0} & \downarrow \\ \text{Inn}(F_N) & \longrightarrow & \text{Aut}(F_N) & \longrightarrow & \text{Out}(F_N) \\ \downarrow & & \downarrow & & \downarrow \\ \pi & \longrightarrow & \prod_{SK_1} G_i & \longrightarrow & \prod G_i \\ \downarrow & & \downarrow & & \downarrow \rho_K \\ \text{Inn}(\pi) & \longrightarrow & \text{Aut}(\pi) & \longrightarrow & \text{Out}(\pi). \end{array}$$

where the dotted homomorphisms are yet to be determined if they exist. The goal is to show that if there is a homomorphism $r : \text{Out}(F_N) \rightarrow \text{Out}(\pi)$ induced by $p_{\#}$, then there is a homomorphism $\rho_K : G_1 \times \cdots \times G_n \rightarrow \text{Out}(\pi)$ such that the following diagram commutes

$$\begin{array}{ccc} & & \text{Out}(F_N) \\ & \nearrow \rho_{K_0} & \downarrow r \\ G_1 \times \cdots \times G_n & & \text{Out}(\pi). \\ & \searrow \rho_K & \end{array}$$

That means, $\rho_K = r \circ \rho_{K_0}$. Hence, we want to find such a map r . ■

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